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Wedderburn Theorem on Varieties of Algebras

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In this paper we extend a recent result of Liu ([6]) to a larger class of algebras which includes all well-known algebras. The main theorem here states: if the Wedderburn–Malcev theorem (it is called the Levi–Malcev–Harish–Chandra theorem in Lie algebra over a field of characteristic 0) holds for a certain variety of finite-dimensional algebra, then it holds for the same variety of algebras which are local subideal finite. We do not use the trace argument in this paper as which was heavily depended on in ([6]). Hence the result is independent of the characteristics of the ground field F.

1. LOCAL SOLVABLE RADICAL

In this paper, we study varieties of algebras which satisfy a certain set of identities (such as associative algebras, Lie algebras, standard algebras, etc.). We do not limit the number of identities. However we do require that all identities be homogeneous. Those algebras under discussion need not be finite dimensional. But they are locally finite; that is to say, any subset X of an algebra A generates a finite-dimensional subalgebra $\langle X\rangle$.

All algebras are considered to be algebras over a fixed field F. Let B be a subalgebra of A. We define $B^{(k)}$ inductively on k by setting $B^{(0)}=B$ and $B^{(k+1)}=B^{(k)}\cdot B^{(k)}$. Thus, for any subalgebra B, we have a descending sequence $B=B^{(0)}\geqslant B^{(1)}\geqslant\cdots\geqslant B^{(n)}\ldots$. We note that, for each i, $B^{(i+1)}$ is an ideal of $B^{(i)}$, hence each $B^{(i)}$ is a subideal of B. A subalgebra B is called a solvable subalgebra if $B^{(k)}=0$ for some integer k. An algebra is locally solvable if any finite subset of A generates a solvable subalgebra. It is easy to see that solvable algebras are locally solvable and homomorphic images of a locally solvable algebra are locally solvable.

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A subalgebra B of A is called a local subideal if, for any finite subset X of A, B is a subideal of $\langle B, X\rangle$, the subalgebra of A generated by B and X. It is clear than any subideal of A is a local subideal of A.

THEOREM 1. *For any local finite algebra A there exists a unique maximal locally solvable ideal $R = R(A)$. The quotient algebra $\bar{A} = A/R$ is semi-simple. That is A/R contains no non-zero locally solvable ideal.*

The proof of this theorem is rather routine. In fact it is exactly the same proof for the existence of the Leviztki radical for associative algebras. Thus we shall omit the proof. As usual we shall call the ideal $R = R(A)$ in Theorem 1 the radical of A. In the sequel, we shall restrict ourself to study varieties of algebras in which solvable radical property for finite-dimensional algebras is weak hereditary. Thus, if B is an ideal of a finite-dimensional algebra A, then $R(B)$ is contained in $R(A)$.

THEOREM 2. *If a weak hereditary property for finite-dimensional algebras holds in the variety $\mathscr{V}$ and if B is a finite-dimensional local subideal of an algebras A in $\mathscr{V}$, then $R(B)$ is contained in $R(A)$.*

Proof. If B is semi-simple, then $R(B) = 0$. There is nothing to be shown. We shall assume that B is not semi-simple. Hence B possesses a non-zero solvable ideal C.

Let C_A be the ideal of A generated by C. If $X = \{x_1, x_2, ..., x_n\}$ is a subset of C_A, then each x_i is a finite sum of elements of the form $T_{a_1} T_{a_2} \cdots T_{a_k}$, where a_j is in A. T_u is either the right or left multiplication of u. The set Y which consists of all a's appears as factors in some x_i is a finite subset of A. The subalgebra $D = \langle B, Y\rangle$ generated by B and Y is a finite-dimensional subalgebra of A. The algebra B is a subideal of D. This is so because that B is a finite-dimensional local ideal of A, Y is a finite subset of A and A is locally finite. Now we see that C is a solvable ideal of B implies that C is contained in $R(B)$. Hence, by the weak hereditary property, C is contained in $R(D)$. The set, C_D, which is the ideal of D generated by C, is an ideal of $R(D)$. This shows that C_D is a solvable ideal in $R(D)$. Since X is contained in C_D. The subalgebra $\langle X\rangle$ is solvable. This proves that C_A is a locally solvable ideal of A containing C, and that $R(B)$ is contained in $R(A)$.

COROLLARY. *If weak hereditary property for finite-dimensional algebras holds in the variety A and if B is a finite-dimensional local subideal of an algebra A, then B is semi-simple if A is.*

2. n-VARIETIES

If n is a positive integer greater than 1, we call a (homogeneous) variety $\mathscr{V}$ an n-variety if whenever A is in $\mathscr{V}$ and I is an ideal of A, then I^n is an ideal of A. The set I^n consists of all finite sums of elements of the form $x_1 x_2 x_3 \cdots x_n$, where x_i is in I and the product could be of any association. It is well known that Lie, associative, alternative and (γ, δ)-algebras are 2-algebras while Jordan, standard and Thedy algebras are 3-algebras. We do not know any n-algebra (which is not a k-algebra with $k < n$) for n greater than 3. Thus, in the sequel we shall only study n-varieties for $n = 2$ or 3.

Anderson has shown us (see [1, 2]) that if A is a 2-algebra, then for any y, x_1, x_2 in A, there exists 16 constants $\alpha_1, \ldots, \alpha_8, \beta_1, \ldots, \beta_8$ such that

$$\begin{aligned} y(x_1 x_2) = {} & \alpha_1 (yx_1) x_2 + \alpha_2 (yx_2) x_1 + \alpha_3 x_1 (yx_2) \\ & + \alpha_4 x_2 (yx_1) + \alpha_5 x_1 (yx_2) + \alpha_6 x_2 (yx_1) \\ & + \alpha_7 x_1 (x_2 y) + \alpha_8 x_2 (x_1 y); \end{aligned} \tag{1_2}$$

$$\begin{aligned} (x_1 x_2) y = {} & \beta_1) yx_1) x_2 + \beta_2 (yx_2) x_1 + \beta_3 x_1 (yx_2) \\ & + \beta_4 x_2 (yx_1) + \beta_5 x_1 (yx_2) + \beta_6 x_2 (yx_1) \\ & + \beta_7 x_1 (x_2 y) + \beta_8 x_2 (x_1 y). \end{aligned} \tag{2_2}$$

Using the same argument as Anderson, we consider the free A algebra generated by $\{x_1, x_2, x_3, y\}$ and consider the ideal D generated by $\{x_1, x_2, x_3\}$. If it is a 3-variety, then we have D^3 is an ideal of A. Thus we have a similar but more tedious equations for this algebra. If A is a 3-algebra and I is an ideal of A, then for any y, x_1, x_2, x_3 in A there exists 288 constants $\alpha_1, \alpha_2, \ldots, \alpha_{72}, \beta_1, \beta_2, \ldots, \beta_{72}, \gamma_1, \ldots, \gamma_{72}, \delta_1, \ldots, \delta_{72}$ such that

$$\begin{aligned} y((x_1 x_2) x_3) = {} & \alpha_1 (yx_1)(x_2 x_3) + \alpha_2 (yx_1)(x_3 x_2) + \alpha_3 ((yx_1) x_2) x_3 \\ & + \alpha_4 ((yx_1) x_3) x_2 + \alpha_5 x_2 ((yx_1) x_3) + \alpha_6 x_3 ((yx_1) x_2) \\ & + \alpha_7 (x_2 (yx_1)) x_3 + \alpha_8 (x_3 (yx_1)) x_2 + \alpha_9 (x_2 x_3)(yx_1) \\ & + \alpha_{10} (x_3 x_2)(yx_1) + \alpha_{11} x_2 (x_3 (yx_1)) + \alpha_{12} x_3 (x_2 (yx_1)) \\ & + \alpha_{13} (x_1 y)(x_2 x_3) + \alpha_{14} (x_1 y)(x_3 x_2) + \alpha_{15} ((x_1 y) x_2) x_3 \\ & + \alpha_{16} ((x_1 y) x_3) x_2 + \alpha_{17} x_2 ((x_1 y) x_3) + \alpha_{18} x_3 ((x_1 y) x_2) \\ & + \alpha_{19} (x_2 (yx_1)) x_3 + \alpha_{20} (x_3 (x_1 y)) x_2 + \alpha_{21} (x_2 x_3)(x_1 y) \\ & + \alpha_{22} (x_3 x_2)(x_1 y)(x_1 y) + \alpha_{23} x_2 (x_3 (x_1 y)) + \alpha_{24} x_3 (x_2 (x_1 y)) \\ & + \alpha_{25} x_1 ((yx_2) x_3) + \alpha_{26} x_1 (x_3 (yx_2)) + \alpha_{27} (x_1 (yx_2)) x_3 \\ & + \alpha_{28} (x_1 x_3)(yx_2) + \alpha_{29} (yx_2)(x_1 x_3) + \alpha_{30} x_3 (x_1 (yx_2)) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_{31}((yx_2)x_1)x_3+\alpha_{32}(x_3x_1)(yx_2)+\alpha_{33}((yx_2)x_3)x_1\\
&+\alpha_{34}(x_3(yx_2))x_1+\alpha_{35}(yx_2)(x_3x_1)+\alpha_{36}x_3((yx_2)x_1)\\
&+\alpha_{37}x_1((x_2y)x_3)+\alpha_{38}x_1(x_3(x_2y))+\alpha_{39}(x_1(x_2y))x_3\\
&+\alpha_{40}(x_1x_3)(x_2y)+\alpha_{41}(x_2y)(x_1x_3)+\alpha_{42}x_3(x_1(x_2y))\\
&+\alpha_{43}((x_2y)x_1)x_3+\alpha_{44}(x_3x_1)(x_2y)+\alpha_{45}(x_2y)(x_1x_3)\\
&+\alpha_{46}(x_3(x_2y))x_1+\alpha_{47}(x_2y)(x_3x_1)+\alpha_{48}x_3((x_2y)x_1)\\
&+\alpha_{49}x_1(x_2(yx_3))+\alpha_{50}x_1((yx_3)x_2)+\alpha_{51}(x_1x_2)(yx_3)\\
&+\alpha_{52}(x_1(yx_3))x_2+\alpha_{53}x_2(x_1(yx_3))+\alpha_{54}(yx_3)(x_1x_2)\\
&+\alpha_{55}(x_2x_1)(yx_3)+\alpha_{56}(x_1(yx_3))x_2+\alpha_{57}x_2(x_1(yx_3))\\
&+\alpha_{58}(yx_3)(x_1x_2)+\alpha_{59}x_2((yx_3)x_1)+\alpha_{60}(yx_3)(x_2x_1)\\
&+\alpha_{61}x_1(x_2(x_3y)+\alpha_{62}x_1((x_3y)x_2)+\alpha_{63}(x_1x_2)(x_3y)\\
&+\alpha_{64}((x_3y)x_1)x_2+\alpha_{65}x_2(x_1(x_3y))+\alpha_{66}(x_3y)(x_1x_2)\\
&+\alpha_{67}(x_2x_1)(x_3y)+\alpha_{68}((x_3y)x_1)x_2+\alpha_{69}x_2(x_1(x_3y))\\
&+\alpha_{70}((x_3y)x_2)x_1+\alpha_{71}x_2((x_3y)x_1)+\alpha_{72}((x_3y)x_1)x_2. \qquad (1_3)
\end{aligned}
$$

$$y(x_1(x_2x_3))=\beta_1(yx_1)(x_2x_3)+\cdots+\beta_{72}((x_3y)x_1)x_2. \qquad (2_3)$$

$$((x_1x_2)x_3)y=\gamma_1(yx_1)(x_2x_3)+\cdots+\gamma_{72}((x_3y)x_1)x_2. \qquad (3_3)$$

$$(x_1(x_2x_3))y=\delta_1(yx_1)(x_2x_3)+\cdots+\delta_{72}((x_3y)x_{1)}x_2. \qquad (4_3)$$

From these equations we easily see:

LEMMA A. *If A is a 3-algebra and B is an ideal of A, then $B^2+B^2A+AB^2$ is a 2-sided ideal of A.*

COROLLARY. *If A is a commutative 3-algebra and B is an ideal of A, then B^2+B^2A is an ideal of A.*

Note. For Jordan algebra A (which is a 3-algebra) Penico has shown that B^2+B^2A is an ieal of A if B itself is an ideal of A.

THEOREM 3. *If A is a simple (which means A possesses no non-trivial ideal and $A^2\neq 0$) algebra which has a finite-dimensional local subideal B, then A is contained in B. Hence A is finite dimensional.*

Proof. Since A is semi-simple, by Theorem 2, B is a finite-dimensional semi-simple subalgebra of A. We can find a non-zero subideal $C=B^{(k)}$ for some integer k such that $C=C^2=C^3$.

For any arbitrary x in A, C is a subideal of $\langle B, X\rangle$. Because C is a subideal of B and B is a subideal of $\langle B, x\rangle$. Thus, from those identities for n-algebras ((1_2), (2_2) for 2-algebras; (1_3), (2_3), (3_3) and (4_3) for 3-algebras) we check that C is indeed an ideal of $\langle B, X\rangle$. But this implies $Cx \leqslant C$ and $xC \leqslant C$. Hence C is an ideal of A.

The algebra A is simple and C is a non-zero ideal of A. Thus $A = C$ and is contained in B.

THEOREM 4. *If A is a semi-simple algebra which has a finite-dimensional local subideal B, suppose that B has a minimal ideal C, then C is an ideal of A.*

Proof. Since C is a minimal ideal of B, we conclude that $C = C^2$ if A is a 2-algebra and $C = C^3$ if A is a 3-algebra. The rest of the proof that C is an ideal of A is same as that in last theorem.

3. LOCAL SYSTEM

A local system (see [3]) of an algebra A is a collection $\{A_\lambda/\lambda \in \Lambda\}$ of subalgebras of A such that (1) $\bigcup A_\lambda = A$ and (2) for λ, μ in Λ there exists a σ in Λ such that $\langle A_\lambda, A_\mu\rangle \leqslant A_\sigma$. An algebra A is said to be local subideal finite if it has a local system $\{A_\lambda/\lambda \in \Lambda\}$ such that (1) $\dim_F(A_\lambda) < \infty$ and (2) for each λ, A_λ is a local subideal of A.

THEOREM 5. *Let $\mathscr{V}$ be an n-variety. Assume that in this variety any finite-dimensional algebra is semi-simple if and only if it is a direct sum of simple algebras and solvable radical property is weak hereditary. Then a local subideal finite algebra A in $\mathscr{V}$ is semi-simple if and only if it is a direct sum of simple finite-dimensional algebras.*

Proof. Assume that A is semi-simple. Then, by Theorem 2, each A_λ is semi-simple. But $\dim_F A_\lambda < \infty$; thus, by the assumption, for each λ, A_λ is a direct sum of finite-dimensional simple subalgebras $A_\gamma = \sum_i \oplus A_{\lambda,i}$. Moreover, by Theorem 4, each $A_{\lambda,i}$ is an ideal of A. Thus $A = \sum_{\lambda,i} A_{\lambda,i}$ and by removing duplicated simple components from this sum, $A = \sum \oplus A_{\lambda,i}$.

THEOREM 6. *Any homomorphic image $\bar{A}$ of a local subideal finite algebra A is a local subideal finite algebra. Any subalgebra A' of a local subideal finite algebra A is a local subideal finite algebra.*

Proof. If $\{A_\lambda : \lambda \in \Lambda\}$ is a local subideal system of A, then it is easy to verify that $\{\bar{A}_\lambda : \lambda \in \Lambda\}$ is a local subideal system of $\bar{A}$.

If A' is a subalgebra of A, then for each $\lambda \in \Lambda$, we let $A'_\lambda = A' \cap A_\lambda$. Then it is easy to see that $\{A'_\lambda | \lambda \in \Lambda\}$ is a local system of A. Moreover $\dim A'$ is finite for each λ. If X is a finite subset of A', then, for each $\lambda \in \Lambda$, there is a chain of subideals $A_\lambda \lhd C_1 \lhd C_2 \lhd \cdots \lhd C_r = \langle A_\lambda, X\rangle$. Thus, we have $A' \cap A_\lambda \lhd A' \cap C_1 \lhd A' \cap C_2 \lhd \cdots \lhd A' \cap C_r = A' \cap \langle A_\lambda, X\rangle$. But $\langle A' \cap A_\lambda, X\rangle \leqslant A' \cap \langle A_\lambda, X\rangle$. So $A' \cap A_\lambda = A' \cap A_\lambda \cap \langle A' \cap A_\lambda, X\rangle \lhd A' \cap C_1 \cap \langle A' \cap A_\lambda, X\rangle \lhd \cdots \lhd A' \cap \langle A_\lambda, X\rangle \cap \langle A' \cap A_\lambda, X\rangle = \langle A' \cap A_\lambda, X\rangle$. This shows that A'_λ is a subideal of $\langle A'_\lambda, X\rangle$ and that A' is a local subideal finite algebra.

4. Wedderburn–Malcev-type Theorems

If A is an algebra, then an automorphism is called an inner automorphism if $\Phi = 1 + f(R_{a_i}, L_{b_j})$, where a_i, b_j are elements of A and at least one of them is in R, the radical of A. R_u (respectively L_v) is the right (respectively left) multiplication of u (respectively v) and $f(x_i, y_j)$ is a polynomial in x_i, x_j (non-commutative) without constant term. We also require an inner automorphism Φ to satisfy condition (0): Whenever B is a finite-dimensional algebra in the variety $\mathscr{V}$ containing A, and $R(A) \subseteq R(B)$, then any inner automorphism Φ of A is an automorphism of B.

Inner automorphism of this type are plenty among all well-known varieties of algebras. (See [4].) Here we note:

LEMMA B. *If A is a local finite algebra in $\mathscr{V}$ and B is a finite-dimensional local subideal of A, then any inner automorphism of B is an inner automorphism of A.*

Proof. Let x, y be elements of A and $D = \langle B, x, y\rangle$ be the subalgebra generated by B, x and y. Then B is a subideal of D and $\dim_F D$ is finite. So Φ is an inner automorphism of D. This shows that Φ is in fact an automorphism of A.

An n-variety $\mathscr{V}$ is said to be of WM-type if for any finite dimensional algebra A in $\mathscr{V}$ we have (1) $A = S \oplus R$, where $R = R(A)$ is the solvable radical of A and the sum in the above equation is semi-direct, (2) A is a direct sum o simple subalgebra, i.e., $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, if A is semi-simple, (3) if S_1 is a semi-simple subalgebra of A, then there exists an inner automorphism Φ such that $S_1^\Phi \subseteq S$, and (4) the solvable radical property for finite-dimensional algebras is weak hereditary.

Associative algebras, Lie algebras over fields of characteristic zero, Jordan algebras and alternative algebras are well known to be algebras of WM-type.

THEOREM 7. *If $\mathscr{V}$ is a 2-variety of* WM-*type, then for any local subideal finite algebra A in $\mathscr{V}$ there exists a semi-simple subalgebra S that $A = S \oplus R$, where $R = R(A)$ is the local solvable radical of A.*

Proof. Let $R = R(A)$ be the radical of A. Then $\bar{A} = A/R$ is semi-simple. By Theorem 6, the algebra $\bar{A}$ is local subideal finite. Thus, $\bar{A} = \oplus \sum_{\lambda \in W} \bar{A}$ is a direct sum of simple subalgebras $\bar{A}_\lambda$. Each $\bar{A}_\lambda$ is of finite-dimension but W may be infinite. We first put an well-ordering on W and then we shall show that we can, for each λ in W, pick a simple subalgebra P_λ in A such that P_λ is of finite dimension, $P_\lambda \cap R = 0$, and $\bar{P}_\lambda = \bar{A}_\lambda$. Moreover $P_\lambda \cdot \oplus \sum_{\sigma < \lambda} P_\sigma \cdot P_\lambda$, so $S = \oplus \sum_{\lambda \in W} P_\lambda$ is a semi-simple subalgeba of A such that $A = S \oplus R$ (semi-direct sum).

Let $\{\bar{a}_1, \bar{a}_2, ..., \bar{a}_m\}$ be a basis of $\bar{A}_1$. For each i, let $\bar{a}_i = a_i + R$, where a_i is in A. Pick member B of the local system of A which contains a_i. Since $\bar{A}_1 \subseteq \bar{B}$, $\bar{A}_1 = \bar{A}_1^{(k)} \subseteq \bar{B}^{(k)}$, the algebra B is not solvable. If the chain $B \supseteq B^{(1)} \supseteq B^{(2)} \cdots B^{(k)} = B^{(k+1)} = \cdots$ stabilizes at $B^{(k)=C}$, then C is a non-zero idemportent subalgebra of A, i.e., $C^2 = C$.

We know that C is a subideal of B and B, in turn, is a local subideal of A. So C is an ideal of A (recall that $C = C^2 = C^3 = \cdots$). In summary, we have found a finite-dimensional ideal C of A such that $\bar{A}_1 \subseteq \bar{C} \subseteq \bar{A}$. The algebra $\bar{C}$ is a finite-dimensional semi-simple subalgebra of $\bar{A}$, and the radical $R(C)$ of C is $C \cap R$. Thus we have a semi-simple subalgebra P of C such that $C = R(C) \oplus P$ (semi-direct sum). Therefore there is a subalgebra P_1 of P such that $\bar{P}_1 = \bar{A}_1$ and $P_1 \cap R = P_1 \cap C \cap R = 0$.

If m is a finite ordinal, and for each $r < m$, we have already chosen a finite-dimensional simple subalgebra P_r of A such that $\bar{P}_r = \bar{A}_r$, $P_r \cap A = 0$, and $H = P_1 \oplus \cdots \oplus P_{m-1}$, we shall find P_m, a finite dimensional simple subalgebra of A, such that $P_m \cap R = 0$, $\bar{P}_m = \bar{A}_m$, and $P = P_1 \oplus \cdots \oplus P_{m-1} \oplus P_m$ is a direct sum.

Let $\{\bar{b}_1, \bar{b}_2, ..., \bar{b}_n\}$ be a basis of $\bar{A}_m$. If, for each i, let $\bar{b}_i = b_i + R$ and B is a member in the local system of A containing all b's. Then as we have shown above, B is a non-solvable subalgebra of A and there exists a non-zero idempotent ideal C of A such that $\bar{A}_m \subseteq \bar{C} \subseteq \bar{A}$. The algebra $\bar{C}$ is a finite-dimensional semi-simple subalgebra of $\bar{A}$ and the radical $R(C)$ of C is $R(C) = C \cap R$. Thus, we have a semi-simple subalgebra Q of C such that $C = R(C) \oplus Q$ (semi-direct sum). Therefore, Q contains a subalgebra Q_m such that $\bar{Q}_m = \bar{A}_m$ and $Q_m \cap R = Q_m \cap C \cap R = 0$.

If $HQ_m = Q_m H = 0$, then $P = P_1 \oplus \cdots \oplus P_{m-1} \oplus Q_m$ is a finite-dimensional subalgebra which we are looking for. Suppose, this is not the case: we may find a member B in the local system of A which contains $Q_m, P_1, ..., P_{m-1}$. Again, B contains a finite-dimensional idempotent ideal D and $D = R(D) + S$, where S is a semi-simple subalgebra of D. Because that $\bar{D}$ contains $\bar{A}_1 \oplus \bar{A}_2 \oplus \cdots \oplus \bar{A}_{m-1} \oplus \bar{A}_m$, we have $S_1, S_2, ..., S_m$, each of this is a simple subalgebra of D such that $\bar{S}_i = \bar{A}_i$, $S_i \cap R = 0$, and $P = S_1 \oplus \cdots \oplus S_m$ is a direct summand of S.

If C is a finite-dimensional ideal of an algebra A and H is a subalgebra of A, then $R_H = \{x \text{ in } H/Cx = 0\}$ and $L_H = \{x \text{ in } H/xC = 0\}$ are subspaces of

H each of finite co-dimension. Let $K = R_H \cap L_H$. Then K is a subspace of H of finite co-dimension. Thus we have $H = K \oplus K^*$, where the sum is the direct sum of vector spaces and K^* is of finite dimension.

Now we are ready to show the theorem by transfinite induction. If σ is an ordinal such that, for each $\lambda < \sigma$, there exists a subalgebra P_λ of A_λ such that $\bar{P}_\lambda = \bar{A}_\lambda$, $P_\lambda \cap R = 0$ and $H = \sum_{\lambda<\sigma} \oplus P_\lambda$ (direct sum of subalgebras). Then as we have shown for $\bar{A}_1$ there exists a finite-dimensional ideal C_σ of A such that $C_\sigma^2 = C_\sigma$, $\bar{C}_\sigma = \bar{A}_\sigma$, $C_\sigma = (C_\sigma \cap R) \oplus P'$ (semi-direct sum), $P' = P'_{\sigma,1} \oplus P'_{\sigma,2} \oplus \cdots \oplus P'_{\sigma,r}$ where, in particular, we let $\bar{P}'_{\sigma,1} = \bar{A}_\sigma$. By the last paragraph, the subalgebra H has the decomposition $H = K \oplus K^*$. But A is a 2-variety and C is an ideal of A, we see easily that K is an ideal of H. Thus, the $H = H \oplus K^*$ is a direct sum of ideals, noting that H is a direct sum of simple subalgebras. Moreover, $K^* = P_1 \oplus \cdots \oplus P_K$, by reindexing.

Let B be an member of the local system of A containing C, $P_1, ..., P_k$. Since B cntains a finite-dimensional idempotent ideal D. Moreover $D = R(D) + P''$, where P'' is a semi-simple subalgebra of D, and $P'' = P''_1 \oplus P''_2 \oplus \cdots \oplus P''_k \oplus P''_\sigma \oplus P''_\beta \oplus \cdots \oplus P''_\alpha$, where $\bar{P}''_i = \bar{P}_i = \bar{A}_i$, and $\bar{P}''_\sigma = \bar{P}'_\sigma = \bar{A}_\sigma$. Since K^* is a semi-simple subalgebra of D, there exists an inner automorphism Φ such that $K^*\Phi^{-1} \subseteq p''$. Thus after rearranging the index $P''_i\Phi = P_i$ for $i = 1, 2, ..., k$. Let $P = P''\Phi = P''_1\Phi \oplus P''_2\Phi \oplus \cdots \oplus P''_k\Phi \oplus P''_\sigma\Phi \oplus P''_\beta\Phi \oplus \cdots \oplus P''_\alpha\Phi = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus P_\sigma \oplus P'''_\beta\Phi \oplus \cdots \oplus P''_\delta\Phi$, where $P_\sigma = P''_\sigma\Phi$. We see that $\bar{P}''_\sigma = \bar{P}_\sigma = \bar{A}_\sigma$, P_σ is semi-simple, $P_\sigma \cap R = 0$, and $P_\sigma \cdot P_i = P_i \cdot P_\sigma = 0$ for $i = 1, 2, ..., k$. On the other hand, recognizing the expression formula of the inner automorphism Φ we have $P_\sigma = P''_\sigma\Phi$ is contained in the ideal C which contains P''_σ. Hence P_σ is contained in the ideal C which contains P''_σ. Hence P_σ is contained in C and so $P_\sigma \cdot K = K \cdot P_\sigma = 0$. Thus, $H \cdot P_\sigma = P_\sigma \cdot H = 0$. So $H^* = H + P$ is the direct sum $\sum_{\lambda \leqslant \sigma} \oplus P_\lambda$ of simple subalgebras such that $\bar{P}_\lambda = \bar{A}_\lambda$ for $\lambda \leqslant \sigma$. This completes the induction proof.

Remark. We would like to see Theorem 7 holds true for 3-algebras. However, there is a little hurdle we do not know how to overcome.

In the transfinite induction part of the proof of Theorem 7, the annihilator K of the ideal C is indeed an ideal of A, if A is a 2-algebra. We are not sure this is the case for 3-algebras. Thus, we shall say a 3-variety $\mathscr{V}$ satisfies condition (T) if whenever A is in $\mathscr{V}$ and I is an ideal of A such that $I^2 = I$, then $K = \operatorname{Ann} C = \{x \in A \mid xC = Cx = 0\}$ is an ideal of A. We see that this condition holds for most well-known varieties. For example the identity $(xy)(uv) = -(x \cdot u \cdot (y \cdot v) - (x \cdot v) \cdot (y \cdot u) + [(x \cdot u) \cdot y] \cdot v + [(x \cdot v) \cdot y] \cdot u + [(u \cdot v) \cdot y] \cdot x$ for Jordan algebra yields condition (T). (Let u, v be taken from C and y from K.) So the variety $\mathscr{V}$ of Jordan algebras is a 3-variety satisfies (T).

THEOREM 8. *If $\mathscr{V}$ is a 3-variety of* WM-*type satisfying condition* (T), *then for any local subideal finite algebra A in $\mathscr{V}$ there exists a semi-simple subalgebra S such that $A = S \oplus R$, where $R = R(A)$ is the local solvable radical of A.*

THEOREM 9. *If $\mathscr{V}$ is a n-variety of* WM*-type and A is a local subideal finite algebra in $\mathscr{V}$ and $A = R \oplus S$ is a Wedderburn decomposition of A. Then for any finite-dimensional semi-simple subalgebra Q of A, there exists an inner automorphism of the form $\Phi = 1 + f(R_{a_i}, L_{b_j})$ such that $Q\Phi \subseteq S$.*

Proof. Since $\bar{Q} \subseteq \bar{A} = \bar{S} = \oplus \sum_\lambda \bar{P}_\lambda$, where $S = \oplus \sum_\lambda P_\lambda$ and $\dim_F \bar{Q} < \infty$ by Theorem 5, we may choose $P_1, P_2, \ldots, P_n$ from the direct sum of S such that $\bar{Q}$ is contained in $\bar{P}_1 \oplus \bar{P}_2 \oplus \cdots \oplus \bar{P}_n$.

Let B be a member of the local system of A which contains $Q, P_1, \ldots, P_n$. Because B is not solvable, there exists an integer k such that $B^{(k)} = B^{(k+1)} = \cdots$. Again, we see that $C = B^{(k)}$ is a finite-dimension ideal of A containing $Q, P_1, \ldots, P_n$. By the finite dimensionality, $C = R(C) \oplus S^*$, where S^* is a semi-simple subalgebra of C containing $P_1 \oplus P_2 \oplus \cdots \oplus P_n$. So we have an inner automorphism ϕ of C (hence of A) such that $Q\phi \subseteq S^*$. But $\bar{Q}$ is contained in $\bar{P}_1 \oplus \cdots \oplus \bar{P}_n$. Therefore, $Q\Phi \subseteq P_1 \oplus \cdots \oplus P_n$. This completes the proof.

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